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A NOTE ON SCHUR FUNCTIONS

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ABSTRACT

We extend the Vandermonde determinant and then extend the classical presentation of Schur functions as quotients of such determinants.

We prove (Theorems 1, 1' below) two extensions of the Vandermonde determinant (parts (a), (a')), and then deduce some new presentations for Schur functions (parts (b), (b')). The evaluation of the multi-integrals of the form

$$\int_{z_2}^{z_1} dx_1 \int_{z_3}^{z_2} dx_2 \cdots \int_{z_{n+1}}^{z_n} dx_n f(x_1, \ldots, x_n),$$

for various functions $f(x)$, is motivated by [2]. As an application of Theorem 1 we evaluate, in Theorem 2, that integral when $f(x)$ is the skew-symmetric function $a_\alpha(x_1, \ldots, x_n)$ as defined in [1, page 23].

Our notations follow those in [1], and the proof of (b) here is a slight generalization of the proof of [1, (3.4)].

To formulate Theorems 1 and 2, we need the following notations:

Fix $n, r \in \mathbb{N}$, $0 \le r$, $1 \le n$. Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ and denote by B_α the following $n \times n$ matrix (in the complete symmetric functions h_{α_i})

$$B_\alpha = (h_{\alpha_i}(x_j, x_{j+1}, \ldots, x_{j+r}))_{1 \le i,j \le n}.$$

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Recall that $\delta = \delta(n) = (n-1,\ n-2,\ldots,1,0)$, and denote $B_\delta = B_{\delta(n)} = B^{(r)}_{\delta(n)}$.

THEOREM 1: *With the above notations*

(a) $\det(B_\delta) = \Pi_{1\le i<j\le n}(x_i - x_{j+r})$

(b) *Let* $\lambda = (\lambda_1,\ldots,\lambda_n)$ *be any partition, then* $\det(B_\delta)$ *divides* $\det(B_{\lambda+\delta})$, *and* $[\det(B_\delta)]^{-1}\cdot[\det(B_{\lambda+\delta})] = s_\lambda(x_1,x_2,\ldots,x_{n+r})$ *(the corresponding Schur function).*

THEOREM 2: *Let* $\lambda = (\lambda_1,\ldots,\lambda_n)$ *be a partition (of at most* n *parts), then*

$$\int_{z_2}^{z_1} dx_1 \int_{z_3}^{z_2} dx_2 \cdots \int_{z_{n+1}}^{z_n} dx_n \left[\prod_{1\le i<j\le n}(x_i - x_j)\right]\cdot s_\lambda(x_1,\ldots,x_n) =$$
$$\frac{1}{(\lambda_1+n)(\lambda_2+n-1)\cdots(\lambda_n+1)}\cdot \prod_{1\le i<j\le n+1}(z_i - z_j)\cdot s_\lambda(z_1,\ldots,z_{n+1}).$$

Note: If $r=0$, then $h_{\alpha_i}(x_j) = x_j^{\alpha_i}$, and the above Theorem 1 becomes

(a) the classical Vandermonde determinant, and

(b) the classical presentation of the Schur function $s_\lambda(x_1,\ldots,x_n)$ [1, (3.1)].

Part (a), which we prove first, is a consequence of Lemma 5 below. The proof of (b) is then a slight extension of the argument in [1, page 25].

LEMMA 3: *Let* $0\le s,\ \ 1\le u<v$ *integers, then*

$$h_s(x_1,\ldots,x_v) = \sum_{j=0}^{s} h_j(x_1,\ldots,x_u)h_{s-j}(x_{u+1},\ldots,x_v).$$

Proof: Obvious from the definition of the functions $h_s(x_1,\ldots x_v)$. ∎

COROLLARY 4: *When* $u = v-1$, *we have*

$$h_s(x_1,\ldots,x_v) = \sum_{j=0}^{s} h_j(x_1,\ldots,x_{v-1})\cdot x_v^{s-j}.$$

By symmetry, we also have

$$h_s(x_1,\ldots,x_v) = \sum_{j=0}^{s} x_1^j h_{s-j}(x_2,\ldots,x_v).$$

LEMMA 5: *Let $s \geq 1$, then*

$$h_s(x_1,\dots,x_v) - h_s(x_2,\dots,x_{v+1}) = (x_1 - x_{v+1})h_{s-1}(x_1,\dots,x_{v+1})$$

(this extends the well known identity $x_a^s - x_b^s = (x_a - x_b)h_{s-1}(x_a, x_b)$).

Proof: It easily follows by the above 3 and 4. ∎

6. The proof of 1(a): By induction on n. Note that the last row of $B^{(r)}_{\delta(n)}$ consists of all 1's. Subtracting the j^{th} column from the $j-1$ column, $j = 2,\dots,n$, we obtain :

$$\det(B^{(r)}_{\delta(n)}) = \prod_{i=1}^{n-1}(x_i - x_{i+1+r}) \cdot \det(B^{(r+1)}_{\delta(n-1)})$$

from which the proof follows by induction. ∎

We turn now to the proof of (b), following [1, page 25].

Definition 7: Let $\alpha = (\alpha_1,\dots,\alpha_n) \in \mathbb{N}^n$, then define the following three $n \times n$ matrices H_α, B_α and $M^{(r)}$:

$$H_\alpha(x_1,\dots,x_{n+r}) = (h_{\alpha_i - n + j}(x_1,\dots,x_{n+r}))_{1\leq i,j\leq n},$$
$$B_\alpha = (h_{\alpha_i}(x_j, x_{j+1},\dots,x_{j+r}))_{1\leq i,j\leq n}$$

and $M^{(r)} = ((-1)^{n-i}\, e^{(k,k+1,\dots,k+r)}_{n-i})_{1\leq i,k\leq n}$.

Here $e^{(k,\dots,k+r)}_\ell(x) = e_\ell(x_1,\dots,x_{k-1},x_{k+r+1},\dots,x_{n+r})$ is the ℓ-th elementary symmetric function in the $n-1$ variables $x_1,\dots,x_{k-1},x_{k+r+1},\dots,x_{n+r}$.

LEMMA 8 (analogue of [1, (3.6)]): *With the above notations $B_\alpha = H_\alpha \cdot M^{(r)}$.*

Proof: Denote

$$E^{(k,\dots,k+r)}(t) = \prod_{\substack{i=1 \\ i\neq k,\dots,k+r}}^{n+r} (1 + x_i t) = \sum_{\ell=0}^{n-1} e^{(k,\dots,k+r)}_\ell(x)\cdot t^\ell,$$

and $H(t) = \Pi_{i=1}^{n+r}(1 - x_i t)^{-1} = \sum_{j=0}^{\infty} h_j(x_1,\dots,x_{n+r})t^j$. Clearly,

$$H(t)\cdot E^{(k,\dots,k+r)}(-t) = \Pi_{i=k}^{k+r}(1 - x_i t)^{-1} = \sum_{j=0}^{\infty} h_j(x_k,\dots,x_{k+r})t^j.$$

Equating the coefficients of t^q, we have:

$$\Sigma_u h_u(x_1,\dots,x_{n+r})(-1)^{q-u} e^{(k,\dots,k+r)}_{q-u}(x) = h_q(x_k,\dots,x_{k+r}).$$

Substituting $q = \alpha_i$ and $u = \alpha_i - n + j$ (so $q - u = n - j$), we obtain

$$\sum_{j=1}^{n} h_{\alpha_i - n + j}(x_1, \ldots, x_{n+r})(-1)^{n-j} e_{n-j}^{(k,\ldots,k+r)}(x) = h_{\alpha_i}(x_k, \ldots, x_{k+r}).$$

In matrix form, the equation is $B_\alpha = H_\alpha \cdot M^{(r)}$. ∎

9. The proof of 1(b): Take determinants in Lemma 8.

$$\det(B_\alpha) = \det(H_\alpha) \cdot \det(M^{(r)}).$$

If $\alpha_i = \lambda_i + n - i$ (i.e., $\alpha = \lambda + \delta$), then $\alpha_i - n + j = \lambda_i - i + j$, hence, by [1, (3.4)], $\det(H_{\lambda+\delta}) = s_\lambda(x_1, \ldots, x_{n+r})$ is the corresponding Schur function. In particular, $\det(H_\delta) = s_{\mathbf{0}}(x) = 1$, hence, by setting $\alpha = \delta$ (so $\lambda = \mathbf{0}$), we deduce that $\det(M^{(r)}) = \det(B_\delta) \underset{(a)}{=} \Pi_{1 \le i < j \le n}(x_i - x_{j+r})$.

Setting $\alpha = \lambda + \delta$, we obtain

$$s_\lambda(x_1, \ldots, x_{n+r}) = \det(H_{\lambda+\delta}) = \frac{\det(B_{\lambda+\delta})}{\det(B_\delta)}.$$

∎

In the theory of symmetric functions, there is a duality between the h_λ's and the $e_{\lambda'}$'s. Since some of the above functions involved only part of the variables, the conjugation operator w [1, page 14] cannot be applied here.

However, dualizing $1 + xt \longleftrightarrow (1 - xt)^{-1}$ and $h \longleftrightarrow e$, one follows step by step the previous argument, proving

THEOREM 1': *Let $n - 2 \le r$ and denote*

$$C_\alpha = (e_{\alpha_i}(x_j, \ldots, x_{j+r}))_{1 \le i,j \le n}.$$

Then

(a') $\det(C_\delta) = \Pi_{j=1}^{n-1} \Pi_{i=j}^{n-1}(x_i - x_{r+2+i-j})$

(b') Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be any partition, then $\det(C_\delta)$ divides $\det(C_{\lambda+\delta})$, and $[\det(C_\delta)]^{-1} \cdot [\det(C_{\lambda+\delta})] = s_{\lambda'}(x_1, \ldots, x_{n+r})$.

We sketch the proof!

Follow 3, 4 and 5 with $\Pi_{i=1}^{v}(1 + x_i t)$ (and e_j's) replacing $\Pi_{i=1}^{v}(1 - x_i t)^{-1}$ (and h_j's), to deduce

$$4' : e_s(x_1, \ldots, x_v) - e_s(x_2, \ldots, x_{v+1}) = (x_1 - x_{v+1}) e_{s-1}(x_2, \ldots, x_v).$$

The rest is identical to the proof of (a).

To prove (b'), follow "conjugated" 7 and 8:

Define $E(t) = \Pi_{i=1}^{n+r}(1 + x_i t)$,

$$H^{(k,\dots,k+r)}(t) = \prod_{\substack{i=1 \\ i\neq k,\dots,k+r}}^{n+r} (1 - x_i t)^{-1}.$$

Then $E(t) \cdot H^{(k,\dots,k+r)}(-t) = \Pi_{i=k}^{k+r}(1 + x_i t)$.

Let $E_\alpha(x_1, \dots, x_{n+r}) = (e_{\alpha_i - n+j}(x_1, \dots, x_{n+r}))_{1\le i,j\le n}$ and $N^{(r)} = ((-1)^{n-j} h_{n-j}^{(k,\dots,k+r)}(x))_{1\le i,j\le n}$ (where $h_q^{(k,\dots,k+r)}(x) = h_q(x_1, \dots, x_{k-1}, x_{k+r+1}, \dots, x_{n+r})$. Following ("conjugated") 7 and 8, we obtain that $C_\alpha = E_\alpha N^{(r)}$.

Follow the argument in 9 now, but apply [1, (3.5)] (instead of (3.4)), to conclude the proof of (b').

Various extensions of 1, 1' exist. For example, let

$$B_\alpha(y, x) = (h_{\alpha_i}(y_1, \dots, y_d, x_j, \dots, x_{j+r}))_{1\le i,j\le n},$$
$$H_\alpha(y, x) = (h_{\alpha_i - n+j}(y_1, \dots, y_d, x_1, \dots, x_{n+r}))_{1\le i<j\le n},$$

and $M^{(r)} = M^{(r)}(x)$ as in 1. Then $B_\alpha(y, x) = H_\alpha(y, x) \cdot M^{(r)}(x)$, which implies that

(a) $\det B_\delta(y, x) = \det M^{(r)}(x)$ (Vandermonde!) and

(b) a presentation of $s_\lambda(y_1, \dots, y_d, x_1, \dots, x_{n+r})$.

We turn now to

The proof of Theorem 2: By [1, (3.1)],

$$\begin{aligned} L &= \prod_{1\le i<j\le n} (x_i - x_j) \cdot s_\lambda(x_1, \dots, x_n) = a_\delta \cdot s_\lambda \\ &= \det(x_i^{\alpha_j})_{1\le i,j\le n} \qquad (\alpha_j = \lambda_j + n - j) \\ &= \sum_{\sigma\in S_n} sgn(\sigma) x_1^{\alpha_{\sigma(1)}} \cdots x_n^{\alpha_{\sigma(n)}} \qquad (S_n \text{ is the symmetric group}). \end{aligned}$$

Denote $\int_{z_2}^{z_1} dx_1 \cdots \int_{z_{n+1}}^{z_n} dx_n$ by $\int_z dx$. Clearly,

$$\int_z dx[x_1^{\beta_1} \cdots x_n^{\beta_n}] = \frac{1}{(\beta_1+1)\cdots(\beta_n+1)} \cdot$$
$$(z_1 - z_2)(z_2 - z_3) \cdots (z_n - z_{n+1}) \cdot h_{\beta_1}(z_1, z_2) h_{\beta_2}(z_2, z_3) \cdots h_{\beta_n}(z_n, z_{n+1}).$$

Thus

$$\int_z dx\, L = \sum_{\sigma\in S_n} sgn(\sigma) \int_z dx\ x_1^{\alpha_{\sigma(1)}}\cdots x_n^{\alpha_{\sigma(n)}} = L_1 \cdot L_2, \quad \text{where}$$

$$L_1 = \frac{1}{(\alpha_1+1)\cdots(\alpha_n+1)} \cdot (z_1 - z_2)(z_2 - z_3)\cdots(z_n - z_{n+1}) \quad \text{and}$$

$$L_2 = \sum_{\sigma\in S_n} sgn(\sigma) h_{\alpha_{\sigma(1)}}(z_1,z_2) h_{\alpha_{\sigma(2)}}(z_2,z_3)\cdots h_{\alpha_{\sigma(n)}}(z_n,z_{n+1})$$
$$= \det(h_{\alpha_i}(z_j, z_{j+1}))_{1\le i,j\le n}.$$

By Theorem 1 with $r = 1$, $L_2 = \prod_{1\le i<j\le n} (z_i - z_{j+1}) \cdot s_\lambda(z_1,\ldots,z_{n+1})$ and the proof now follows since

$$(z_1 - z_2)\cdots(z_n - z_{n+1}) \cdot \prod_{1\le i<j\le n} (z_i - z_{j+1}) = \prod_{1\le i<j\le n+1} (z_i - z_j).$$

∎

Note: Since the polynomials $\prod_{1\le i<j\le n} (x_i - x_j) s_\lambda(x_1,\ldots,x_n)$ form a basis for the anti-symmetric polynomials, one can now evaluate $\int_z dx$ for any such function.

References

[1] I. G. Macdonald, *Symmetric Functions and Hall Polynomials*, Oxford Univ. Press, Oxford, 1979.

[2] A. Regev, *Yound-derived sequences of S_n-characters*, preprint.